1. Rešiti Klajn-Gordonovu j-nu (za slobodnu česticu)

$$\left(\Box + \frac{m_0^2 c^2}{\hbar^2}\right)\phi(x) = 0 \qquad [m_0 = m] \qquad (1)$$

Partikularno rešenje (1) tražimo u obliku ravnog talasa

$$e^{-ik\cdot x} = e^{-\frac{i}{\hbar} p_\mu x^\mu} = e^{-\frac{i}{\hbar}(p_0 x^0 - \vec{p}\cdot\vec{x})}$$

$$= e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)} \qquad (2)$$

Provera

$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial x^{0\,2}} - \vec{\nabla}^2$$

$$\left(\frac{\partial^2}{\partial x^{0\,2}} - \vec{\nabla}^2 + \frac{m^2c^2}{\hbar^2}\right) e^{\frac{i}{\hbar}(\vec{p}\,\vec{x} - p_0 x^0)} = 0$$

$$e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - p_0 x^0)} \left(\frac{i}{\hbar}p_0\right)^2 - e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - p_0 x^0)}$$

$$\left\{\left(\frac{i}{\hbar}p_1\right)^2 + \left(\frac{i}{\hbar}p_2\right)^2 + \left(\frac{i}{\hbar}p_3\right)^2\right\} + \frac{m^2c^2}{\hbar^2} e^{\frac{i}{\hbar}(\vec{p}\vec{x} - p_0 x^0)} = 0$$

$$\left(-\frac{p_0^2}{\hbar^2} + \frac{p_1^2}{\hbar^2} + \frac{p_2^2}{\hbar^2} + \frac{p_3^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2}\right) e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - p_0 x^0)} = 0$$

$$p_0 = \frac{E}{c}$$

$$-p_0^2 + \vec{p}^{\,2} + m^2c^2 = 0$$

$$-\frac{E^2}{c^2} + \vec{p}^{\,2} + m^2c^2 = 0 \quad / \cdot c^2$$

$$-E^2 = -\vec{p}^{\,2}c^2 - m^2c^4 \quad , \text{ tj}$$

$$\boxed{E^2 = c^2\vec{p}^{\,2} + m^2c^4}$$

Dakle, $e^{-ik\cdot x}$ jeste rešenje K-G j-ne.

Ali

$$E = \pm\sqrt{c^2\vec{p}^{\,2} + m^2c^4}$$

Postoje i negativna rešenja!

Za fiksnu vrednost vektora $\vec{k}$, postoje dva linearno nezavisna rešenja (B. Dobsaci)

$$e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} \quad i \quad e^{i\omega_k t + i\vec{k}\cdot\vec{x}}$$

Opšte rešenje j-ne (I) je

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left(a(\vec{k})\, e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + b^{+}(-\vec{k})\, e^{i(\omega_k t + \vec{k}\cdot\vec{x})} \right)$$

Pogledati Mušicki III/2 (44.14)

$\left.\begin{array}{l} a(\vec{k}) \\ b^{+}(\vec{k}) \end{array}\right\}$ Kompleksni koeficijenti

Uz zamjenu

$$\vec{k} \to -\vec{k}$$

u drugom članu

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left(a(\vec{k}) e^{-ik\cdot x} + b^{+}(\vec{k}) e^{ik\cdot x} \right)$$

postiže se kompaktniji zapis!

Решења о којима је реч одговарају својственом проблему хамилтонијана, тј. чине св. базис (тј. одговарају позитивној и негативној енергији)

$$\int e^{-ikx} e^{ikx'} d^4k = \int e^{ik(x'-x)} d^4k$$

$$= (2\pi)^4 \delta^4(x'-x)$$

Како је у питању ОНБ $\Rightarrow$ у питању је ЛНЗ скуп

ЛНЗ скупа ф-ја се може утврдити и преко Вронскијана

$$\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = \begin{vmatrix} e^{ikx} & e^{-ikx} \\ ike^{ikx} & -ik\,e^{-ikx} \end{vmatrix} = 2ik \neq 0$$

$k \neq 0$

$$\boxed{\begin{array}{c} W = 0 \not\Rightarrow ЛЗ \\ ЛЗ \Rightarrow W = 0 \end{array}}$$

Data je struja

$$j_\mu = i\,(\phi\,\partial_\mu \phi^* - \phi^* \partial_\mu \phi)$$

Pokazati da ona zadovoljava j-nu kontinuiteta $\partial^\mu j_\mu = 0$. $\underline{\phi = \phi(x)}$ se biće KG j-ne.

$$\partial^\mu j_\mu = i\,(\partial^\mu \phi\,\partial_\mu \phi^* + \phi\,\partial^\mu\partial_\mu \phi^* -$$

$$\partial^\mu \phi^* \partial_\mu \phi - \phi^* \partial^\mu \partial_\mu \phi)$$

$$= i\,(\partial^\mu \phi\,\partial_\mu \phi^* + \phi\,\Box\,\phi^* - \partial^\mu \phi^* \partial_\mu \phi - \phi^* \Box\,\phi)$$

$$(\Box + m^2)\,\phi(x) = 0$$

$$(\Box + m^2)\,\phi^*(x) = 0$$

$$\boxed{\partial^\mu j_\mu = 0} \qquad \underline{j_\mu = (c\rho, -\vec{j})}$$

Ali,

$$j_0 = i\,(\phi\,\partial_0 \phi^* - \phi^* \partial_0 \phi) = c\rho$$

Ovde ρ može biti i negativno, nasuprot nerelativističkoj ~~k~~vantnoj mehanic[i] gde je $\rho = |\psi|^2 \geq 0$.

Razlog: KG jednačina je drugog reda po vremenu!

3. Koristeći kanonske komutacione relacije

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = [\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{y},t)] = 0$$

izvesti sledeće komutacione relacije za kreacione i anihilacione operatore

$$[a(\vec{k}), a^{+}(\vec{q})] = \delta^{3}(\vec{k}-\vec{q})$$

$$[a(\vec{k}), a(\vec{q})] = [a^{+}(\vec{k}), a^{+}(\vec{q})] = 0$$

Razmatrati slučaj realnog KG polja.

Opšte rešenje KG - j-ne

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left(a(\vec{k}) e^{-ik\cdot x} + b^{+}(\vec{k}) e^{ik\cdot x}\right)$$

Ако је 3d ставити опште решење писмени, у поставку задатка

Za realno polje $a(\vec{k}) = b(\vec{k})$

$$\phi = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left[a(\vec{k}) e^{-ik\cdot x} + a^{+}(\vec{k}) e^{ik\cdot x}\right]$$

$\boxed{\phi(x)}$

$$\pi = \dot{\phi}$$

$$\dot{\phi} = \frac{1}{(2\pi)^{\frac{3}{2}}} i \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \omega_k \left[-a(\vec{k}) e^{-ik\cdot x} + a^{+}(\vec{k}) e^{ik\cdot x}\right]$$

$$\int d^3\vec{r}\,\phi(x)\,e^{-i\vec{k}'\cdot\vec{r}} =$$

$$\int d^3\vec{r}\left[\int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}\left[a(\vec{k})e^{-ik\cdot x}+a^+(\vec{k})e^{ikx}\right]\right.$$

$$e^{-i\vec{k}'\cdot\vec{r}} =$$

$$\int d^3\vec{r}\left[\frac{\int d^3\vec{k}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}\left[a(\vec{k})\,e^{-i(\omega_k t-\vec{k}\cdot\vec{r})}+a^+(k)\,e^{i(\omega_k t-\vec{k}\cdot\vec{r})}\right.\right.$$

$$e^{-i\vec{k}'\cdot\vec{r}} =$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}\int d^3\vec{r}\int d^3\vec{k}\,a(\vec{k})\,e^{-i\omega_k t}\,e^{i\vec{r}\cdot(\vec{k}-\vec{k}')} +$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}\int d^3\vec{r}\int d^3\vec{k}\,a^+(\vec{k})\,e^{i\omega_k t}\,e^{-i\vec{r}\cdot(\vec{k}+\vec{k}')} =$$

Definicija δ fje:

$$\delta(x-\alpha)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\beta(x-\alpha)}\,d\beta \qquad (1D)$$

$$= \frac{1}{(2\pi)^{3/2}}\frac{(2\pi)^3}{\sqrt{2\omega_k}}\int d^3\vec{k}\,a(\vec{k})e^{-i\omega_k t}\delta^{(3)}(\vec{k}-\vec{k}') +$$

$$\frac{1}{(2\pi)^{3/2}}\frac{(2\pi)^3}{\sqrt{2\omega_k}}\int d^3\vec{k}\,a^+(\vec{k})\,e^{i\omega_k t}\,\delta^{(3)}(\vec{k}-(-\vec{k}'))$$

Dakle,

$$\int d^3\vec{r}\, \phi(x)\, e^{-i\vec{k}'\cdot\vec{r}} =$$

$$\frac{(2\pi)^{\frac{3}{2}}}{\sqrt{2\omega_{k'}}}\left[a(\vec{k}')e^{-i\omega_{k'}t} + a^{+}(-\vec{k}')e^{i\omega_{k'}t}\right] \quad (1)$$

Slično, dobija se i za $\dot{\phi}(x)$

$$\int d^3\vec{r}\, \dot{\phi}(x)\, e^{-i\vec{k}'\cdot\vec{r}} = i(2\pi)^{\frac{3}{2}}\sqrt{\frac{\omega_{k'}}{2}}\left[a^{+}(-\vec{k}')e^{i\omega_{k'}t} + a(\vec{k}')e^{-i\omega_{k'}t}\right] \quad (2)$$

Iz (1) i (2) se dobija

$$a(\vec{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega_k}}\int d^3\vec{r}\, e^{ikx}\left[\omega_k\phi(x) + i\dot{\phi}(x)\right]$$

$$a^{+}(\vec{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega_k}}\int d^3\vec{r}\, e^{-ikx}\left[\omega_k\phi(x) - i\dot{\phi}(x)\right]$$

Sada tražimo:

$$\boxed{\begin{array}{l} x = (ct, \vec{r}) \\ y = (ct, \vec{\varrho}) \end{array}}$$

$$[a(\vec{k}), a^{+}(\vec{q})] =$$

$$\frac{i}{2(2\pi)^3}\frac{1}{\sqrt{\omega_k\omega_q}}\int d^3\vec{r}\, d^3\vec{\varrho}\, e^{i(kx-qy)} \quad (x)$$

$$\left(-\omega_k \left[\phi(x), \dot{\phi}(y)\right] + \omega_{k'} \left[\dot{\phi}(x), \phi(y)\right]\right)$$

q ?

$$= \frac{1}{2(2\pi)^3} \frac{1}{\sqrt{\omega_k \omega_{q}}} \int d^3\vec{r}\; e^{i(\omega_k - \omega_{q})t + i(\vec{q} - \vec{k})\vec{r}}$$

$$(\omega_k + \omega_{q}) = \underline{\underline{\delta^3(\vec{k} - \vec{q})}}$$

<u>Za domaći ostatak</u>

$$= \underbrace{\frac{1}{2} \frac{1}{\sqrt{\omega_k \omega_q}}\, e^{i(\omega_k - \omega_q)t} (\omega_k + \omega_q)}_{(*)\big|_{\omega_k = \omega_q}} \underbrace{\int \frac{1}{(2\pi)^3} d^3\vec{r}\; e^{i(\vec{q} - \vec{k})\vec{r}}}_{\delta^3(\vec{k} - \vec{q})}$$

$$(*)\; \frac{1}{2} \frac{1}{\omega_k} e^{0}\, 2\omega_k = 1 \quad \checkmark$$

4. Pokazati da za česticu spina nula i naelektrisanja q u spoljnjem EM polju struja glasi:

$$j^\mu = (\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*) + 2iq\, \phi^* A^\mu \phi$$

pri čemu je ϕ rešenje KG j-ne u spoljašnjem EM polju potencijala A^μ.

Minimalna preskripcija

$$D_\mu = \partial_\mu + iqA_\mu$$

$$\Downarrow$$

$$\bar{D}_\mu = \partial_\mu - iqA_\mu$$

$$D^\mu = \partial^\mu + iqA^\mu$$

$$\bar{D}^\mu = \partial^\mu - iqA^\mu$$

$(D_\mu D^\mu - m^2)\phi = 0$ (1). Pa onda $\phi^* / (1)$ a potom $(1)^*$ i $\phi / (1)^*$

$$\Downarrow$$

$$\left.\begin{aligned} \phi^* (D_\mu D^\mu - m^2)\phi &= 0 \\ \phi (\bar{D}_\mu \bar{D}^\mu - m^2)\phi^* &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} \phi^* D_\mu D^\mu \phi &= m^2 \phi^* \phi \\ \phi \bar{D}_\mu \bar{D}^\mu \phi^* &= m^2 \phi \phi^* \end{aligned}\right\} -$$

$$\phi^* D_\mu D^\mu \phi - \phi \bar{D}_\mu \bar{D}^\mu \phi^* = 0$$

$$\phi^*(\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu)\phi - \phi(\partial_\mu - iqA_\mu)(\partial^\mu - iqA^\mu)\phi^* = 0$$

$$\phi^*(\partial_\mu\partial^\mu\phi + iq\,\partial_\mu(A^\mu\phi) + iqA_\mu\partial^\mu\phi - q^2A_\mu A^\mu\phi) -$$
$$-\phi(\partial_\mu\partial^\mu\phi^* - iq\,\partial_\mu(A^\mu\phi^*) - iqA_\mu\partial^\mu\phi^* - q^2A_\mu A^\mu\phi^*) = 0$$

$$\phi^*\partial_\mu\partial^\mu\phi + iq\phi^*(\partial_\mu A^\mu)\phi + iq\phi^*A^\mu\partial_\mu\phi + iq\phi^*A_\mu\partial^\mu\phi -$$
$$-\cancel{q^2\phi^*A_\mu A^\mu\phi} - \phi\partial_\mu\partial^\mu\phi^* + iq\phi(\partial_\mu A^\mu)\phi^* + iq\phi A^\mu\partial_\mu\phi^* +$$
$$+ iq\phi A_\mu\partial^\mu\phi^* + \cancel{q^2\phi A_\mu A^\mu\phi^*} = 0$$

$$(\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^*) + 2iq\phi\phi^*(\partial_\mu A^\mu) + 2iq\phi^*A^\mu\partial_\mu\phi +$$
$$+ 2iq\phi A^\mu\partial_\mu\phi^* = 0$$

$$(\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^*) + 2iq\phi\phi^*(\partial_\mu A^\mu) +$$
$$+ 2iqA^\mu\underbrace{(\phi^*\partial_\mu\phi + \phi\partial_\mu\phi^*)}_{\partial_\mu(\phi\phi^*)} = 0$$

$$(\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^*) + 2iq\phi\phi^*(\partial_\mu A^\mu) + 2iqA^\mu\partial_\mu(\phi\phi^*) = 0$$

$$(\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^*) + 2iq\,\partial_\mu(\phi\phi^*A^\mu) = 0$$

$$\partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + 2iq\,\partial_\mu(\phi\phi^*A^\mu) = 0$$

$$\partial_\mu\Big[\underbrace{(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + 2iq\phi\phi^*A^\mu}\Big] = 0 \qquad j^\mu \text{ (struja)}$$